Mastermath Random Walks 2019 - 2020, Exam 27.01.2020

1. State the local central limit theorem for simple random walk on $\mathbb{Z}^d$ (that is, the walk that jumps from any vertex x to any of the $2d$ neighbors of x with probability $\frac{1}{2d}$). Using this result, prove Pólya's theorem for this walk, that is, prove that the walk is recurrent in dimensions one and two, and transient in dimension three or higher.

2. Suppose that p is the transition function for a random walk on $\mathbb{Z}^d$ that is irreducible, symmetric, has finite range, and satisfies $p(0) = 0$. For $\epsilon > 0$, let p_ϵ denote the increment of the "lazy walker" given by

$$p_\epsilon(x) = \begin{cases} (1-\epsilon)p(x), & x \neq 0, \\ \epsilon, & x = 0. \end{cases}$$

   (a) Show that p_ϵ defines an irreducible and aperiodic random walk on $\mathbb{Z}^d$.

   The main objective of this exercise is to prove that for $d \geq 3$ the Green functions G and G_ϵ, of p and p_ϵ respectively, satisfy the relation

$$G_\epsilon(x) = \frac{1}{1-\epsilon} G(x). \tag{$\star$}$$

   (b) Give a probabilistic proof of ($\star$), by comparing "dynamics" of the lazy walker to that of the original. How can you relate the number of visits to a given site x?

   (c) Give an analytic proof of ($\star$). *Hint: Cha... f....*

3. (a) Let $m, n \in \mathbb{N}$ with $m < n$, and let

$$\mathbb{H}_{m,n} := \{x = (x^1, \dots, x^n) \in \mathbb{Z}^n : x^1 = \dots = x^m = 0\}.$$

   Determine the values of m, n for which simple random walk on $\mathbb{Z}^n$ (started at the origin) visits $\mathbb{H}_{m,n}$ infinitely many times with probability one. Give a proof to justify your answer.

   (b) Define, for $\alpha > 0$,

$$\mathbb{A}^\alpha := \{x = (x^1, x^2, x^3, x^4) \in \mathbb{Z}^4 : x^1 = x^2 = 0,\ |x^4| \leq |x^3|^\alpha\}.$$

   Prove that, if $\alpha < 1$, then simple random walk on $\mathbb{Z}^4$ (started at the origin) only visits $\mathbb{A}^\alpha$ finitely many times, almost surely.

4. (a) State the main coupling theorem for random walks on $\mathbb{Z}^d$ that was proved in the course.

   (b) Prove that if p is the transition function for a random walk on $\mathbb{Z}^d$ that is irreducible and aperiodic, then for any fixed vector $v \in \mathbb{Z}^d$, we can construct, in the same probability space, two processes $(X_n)_{n \geq 0}$ and $(Y_n)_{n \geq 0}$ such that:

   - (X_n) and (Y_n) are p-random walks on $\mathbb{Z}^d$ started at the origin;
   - $X_n - Y_n$ converges in probability to v as $n \to \infty$.

5. Fix $d \in \mathbb{N}$. For each $k \in \mathbb{N}$, let B_k be the subgraph of $\mathbb{Z}^d$ induced by the vertex set $\{-k, \dots, k\}^d$. That is, B_k is the graph with vertex set $\{-k, \dots, k\}^d$ and having all edges $\{x, y\}$ for which $x, y \in \{-k, \dots, k\}^d$ and x, y are neighbors in $\mathbb{Z}^d$. Consider simple random walk $(X_n^{(k)})_{n \geq 0}$ on this graph, that is, the Markov chain whose state space is the set of vertices of B_k and which, from time n to time $n+1$, jumps to a position chosen uniformly at random among all neighbors of the position at time n.

   (a) Let π_k denote the invariant distribution of this walk. Show that for every vertex x of B_k, $\pi_k(x)$ is proportional to the degree of x in B_k.

   (b) Let

$$\mu_{k,n}(x) := \mathbb{P}(X_n^{(k)} = x \mid X_0^{(k)} = 0), \quad \vec{x} \in \{-k, \dots, k\}^d,$$

   that is, $\mu_{k,n}$ is the distribution, at time n, of the walk on B_k started at the origin.

   Prove that, if $(n_k)_{k \in \mathbb{N}}$ is a sequence of positive integers with $\frac{n_k}{k^2} \xrightarrow{k \to \infty} 0$, then $\|\pi_k - \mu_{k,t_k}\|_{\mathrm{TV}} \xrightarrow{k \to \infty} 1$.

6. For each $q \in \mathbb{R}$, $q > 1$, define

$$A_q = \{\lfloor q^n \rfloor : n \in \mathbb{N}\},$$

   where $\lfloor x \rfloor$ denotes the largest integer that is smaller than or equal to x. Let $\mathbb{T}_q$ be a connected tree with the following properties:

   - $\mathbb{T}_q$ has a distinguished vertex o called the root, with degree one (recall that the degree of a vertex is defined as its number of neighbors);
   - for every vertex $x \neq o$, x has degree three (or "two children") if its distance to o belongs to A_q, and x has degree two (or "one child") otherwise.

   For each value of q, determine if simple random walk on $\mathbb{T}_q$ is recurrent or transient.